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How To Improve Pre-Service Mathematics Teachers' Problem-Solving Competencies (Case Study)

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Abstract. Lower and upper secondary school mathematics offers a few tools for solving mathematical problems (e. g., equational strategy or other algorithmic ones). At the beginning of the university study our students (pre-service mathematics teachers) know how to use them, but they have no experience with solving strategies such as systematic experimentation or drawing a picture. This paper presents our lesson plan to improve their solving competencies and its results. The special attention is given to the process of mathematics problem-solving and its fourth phase 'looking back' that we understand in a broader sense, not only as a test of correctness. The results of this case study aim to show and understand how students learn and use various solving strategies in problem-solving and how they pose other word problems with the same solution as the solution of the given word problems. The analysis finds that even though students understand the importance of non-standard strategies (e.g., trial and error or estimation-verification-correction), they still understand the school equational strategy as the most advantageous. For problem-posing to contribute to the improvement of problem-solving competences, it must be more often involved in the problem-solving process.

Keywords. problem-solving competencies, problem-solving process, looking back, solving strategies, problem-posing

1. Introduction

According to (English, & Gainsburg, 2010; Szabo, Körtesi, Guncaga, Szabo, & Neag, 2020), problem-solving is recognized as a part of the 21st-century important life skills. Developing a successful problem solver (students of primary or secondary school) is a process requiring a range of skills and dispositions, e. g., (Stacey, 2005) and leading by a teacher = an expert problem solver.

The process of training pre-service teachers is very complex. It is possible to look at it from the perspective of the TPACK model (e.g., Harris, Mishra, & Koehler, 2009). This model identifies the competencies that a (pre-service) mathematics teacher should have technological, pedagogical, and content knowledge. One of the mathematics teacher's competencies that is at the boundary of pedagogical and content knowledge is problem-solving. Therefore, enough opportunities should be created for teachers to build knowledge about teaching problem-solving and using problems as a focus of learning in mathematics (Cai, 2003).

Our students (pre-service mathematics teachers) usually think they solve mathematics problems at a good level at the beginning of their university studies. However, when they encounter the process of problem-solving and its phases according to G. Polya, they are surprised. In primary and secondary school, they learned many procedures and algorithms for solving problems. (Many of them use one or more equations and no pictures.) But they did not use various strategies for one problem, and they also have not looked for similar problems or posed new problems. In Slovakia, the long-term preferred school approach to problem-solving has got a scheme: "one mathematics problem = only one solving strategy". The solving strategy is chosen according to the topic and given problem and is usually based on a certain algorithm or procedure.[1] The strategy is introduced by the teacher and expected in the student's solution. Solving strategies such as trial and error, estimation-verification-correction, or systematic experimentation (that are very suitable e. g. when it is not clear how to solve a problem) have not been officially used. On the other hand, within the mathematical olympiad (or other mathematics competition for students from lower and upper secondary school), these strategies are allowed, even requested.

However, the complete four-phase process of problem-solving in mathematics according to G. Polya is the result of the work of 'a good problem solver'. Such a solver has a good level of mathematics knowledge and skills, experience with a variety of solving methods and strategies, and insight into which of the strategies is the best or most appropriate to use for the given problem or opportunity. This phenomenon is sometimes encountered when students – pre-service mathematics teachers finish their university studies. But in some cases, mathematics teachers will understand the importance of all four phases of problem-solving only later in their teaching practice.

The aim of this article is to show what approach to problem-solving we have chosen for the beginning of our students' studies at the bachelor's level so that their problem-solving skills improve. We will show what strategies they used when solving word problems and what new problems they posed, and thus they 'passed' the fourth phase of the problem-solving process. We will also publish students' opinions on individual learned and used solving strategies.

2. Process of problem-solving

Problem-solving is understood as central in school mathematics, because "without a problem, there is no mathematics" (Klerlein, Hervey, 2019, p. 1). According to (Evans, 2014), one of the most suitable ways to help students to acquire a better content understanding of the studied topic is by providing them with the opportunity to learn through problem-solving and inquiry learning. Focusing on problem-solving has been a trend in school mathematics in various countries at the end of the 20th century and the beginning of the 21st century.[2]

G. Polya (1945) described problem-solving as a four-phase process (shown in Fig. 1) and emphasized that problem-solving is a practical skill and can be acquired by imitating other

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[1] In the national curriculum for the primary and secondary schools in Slovakia, there is stated that one of the main goals of school mathematics is to use understood strategies and algorithms for solving the problem. However, in our opinion, this goal is not given enough attention.

[2] Slovak curriculum documents present the school mathematics as lists of 'content' elements and 'processes' at each level of ISCED 1, 2, and 3. Content elements include the fundamental concepts of mathematics areas (precisely number theory and arithmetic, algebra, relations and functions, geometry and measurement, and chance and data). Processes include the actions associated with using and applying mathematics to solve problems. In some countries (e. g. Singapore or Finland) the situation is different. Their mathematics curriculum documents focus on problem-solving and related skills.

people and practicing. In the first phase of problem-solving the solver should understand the problem, then make a plan to solve it (phase 2) and execute the prepared plan (phase 3). Finally, the solver should look back and reflect on how the problem has been solved (phase 4). These four phases follow each other, pass continuously from one to the other, and are also interconnected (Klerlein, Hervey, 2019), as it is shown in Fig. 1. In the process of solution, the solver may change his/her point of view several times.

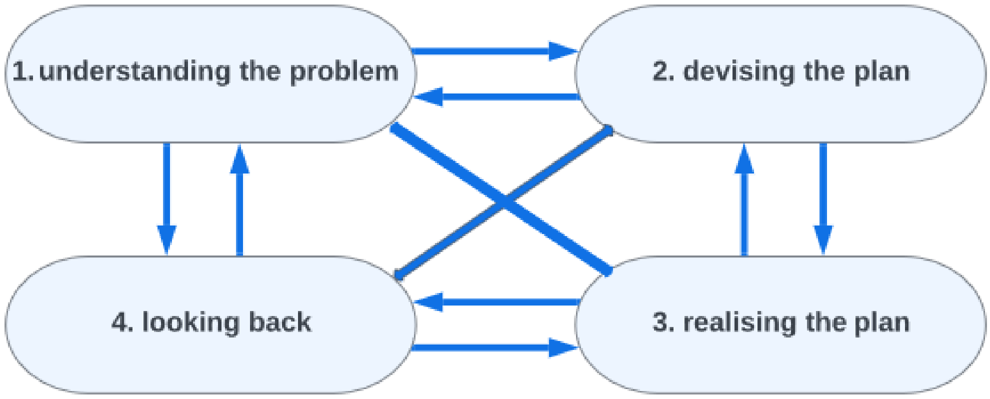

Figure 1. Process of problem-solving with four interconnected and mutually influencing phases (source: authors)

The interconnection of the phases is important not only within one solution cycle. For example, the first phase ‘understanding the problem’ can be understood “as a continuation of all the previous mathematical experience, in which understanding of new problems requires ‘looking back’ at those solved in the past” (Applebaum, Leikin, 2007, p. 258).

2.1. Looking back

In school mathematics, students often end their problem-solving in the third phase when the word answer is written (or the numerical result is underlined). The fourth phase (looking back) is so the most neglected of the four ones (Cai, Brook, 2006; Taback, 1988). It is not just checking the solution and finding another strategy as G. Polya presented. It can be understood as the reflection of learned and done in a learning process in the sense of understanding Dewey or Piaget and the inclusion of looking back strategies can improve student performance (Lee, 2016; Lee, 2021) and encourage students to maximize their learning opportunities (Cai, Brook, 2006).

Cai and Brook (2006) extend Polya’s thoughts and propose three ways of implementing ‘looking back’ at the end of the problem-solving: 1. generating, analysing, and comparing alternative solutions, 2. posing new problems, and 3. generalizing. Within the fourth phase, the students can present and defend their own ideas, strategies, and solutions, and discuss them with colleagues in the class. Extension of this phase by posing new problems can increase students’ understanding of the problem and enhance their skills in problem-solving. In generalization, the students can discover mathematical structures and relationships between concrete and specific. Such an extension of the looking back phase definition seems to be very beneficial for mathematics education.

2.2. **Problem-Solving Strategies**

There are many studies describing problem-solving and solving strategies in the solutions of primary and secondary school students or describing how to solve problems and how to teach problem-solving, e. g. (Polya, 1945), (Schoenfeld, 1980, 1992). (Eisenmann, Přibyl, Novotná, Břehovský, & Cihlář, 2017) found out, that the pupils were able to use some heuristic strategies even though they had not learned them in the mathematics lessons. There is also a complex range of studies that reflect a need to develop mathematics teachers' problem-solving skills and strategies, e. g. (Barham, 2021).

There is no precise professional agreement on what strategy in mathematical problem-solving is, or what exactly is the difference between a solving strategy and a solving method. However, we understand the solving strategy mainly as a way of solving the problem as Czech experts deal with this concept and its content, as Kopka (2010), Novotná, Eisenmann, Přibyl, Ondrušová & Břehovský (2014), or Eisenmann, Přibyl, Novotná, Břehovský, & Cihlář, (2017). Such an approach is sufficient for solving-problem skills of the solver and their improvement.

For our research, we use the following strategies: trial and error, estimation–verification–correction, systematic experimentation, logical reasoning, equational strategy, and drawing a picture.[3] It is not a complete list of problem-solving strategies. That is the list of the strategies for solving the word problems from school mathematics that we use most often.

It is important to stress when solving a problem, we do not use only one "clean" strategy. In most cases, strategies can be combined simultaneously or smoothly transition from one to another.

2.3. **Problem-posing**

Problem-posing can be understood as a part of Polya's last phase 'looking back', as was mentioned above. It is the ability of students to create/pose new problems or reformulate problem assignments that were given or were based on some situation or experience (Silver, 1994, 1995). According to Silver and Cai (1996), problem-posing has received more attention in recent years to improve the students' competencies to solve problems and their skills in creating mathematical problems. Problem-posing also develops creativity, and mathematical understanding and encourages students to autonomous learning (Arikan, Unal, 2013). However, problem-posing not only develops the abilities of students but also of the teachers, especially their competencies, "including their sensitivity to students' mathematical thinking" (Cai, & Leikin, 2020, p. 289). However, the algorithm for students' problem-posing is still not completely clear.

Kilpatrick (in Cai, & Leikin, 2020) stated, the quality of problems posed by students might be the index of the ability of students to solve the problems. This correlation may serve as an inspiration for the next research.

As was stated by Arikan and Unal (2013), problem-posing has been accepted as an important part of mathematics, but the mathematics education community does not focus on this problem. One of the reasons might be the ignorance of the teachers about the problem-posing. This situation is not just in Slovakia, but it appears worldwide.

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[3] There are many other problem-solving strategies identified by researchers, e. g. *looking for a pattern, solving a simpler problem, working backward*, and *using a formula*.

3. Methodology of case study

The paper presents a case study in which students participated in a course focused on repeating the mathematics basics from secondary school and on mathematics problem-solving. This study mainly shows the part of the fourth phase of the problem-solving process ('looking back') of pre-service mathematics teachers from our university in Slovakia (we will call them students). They first had to solve the problems with 3 different strategies and then pose a new word problem.[4]

The lesson plan aiming to improve students' solving competencies is based on our experience (and experience of our cooperated mathematics teachers) that students fixate on the algebraic way of solutions (through equations) and do not know other solving strategies.

On this basis, the research questions on which this research was focused are the following:

1. *What solving strategies do the students use when solving the problems? Do they use also drawing a picture strategy?*
2. *Which of the used solving strategies do the students use as the first one?*
3. *What problems do the students pose? Do new problems reflect the understanding of the given problems and students' solutions?*
4. *Do students consider individual problem-solving strategies useful or suitable for solving problems?*

3.1. **Participants**

The sample of study participants consisted of 10 pre-service mathematics teachers. They attended an introduction course during the first semester of the academic year 2022/23 at the Catholic University in Ružomberok. According to the new accreditation, this course is aimed at the repetition of secondary school mathematics and unifying the level of students' problem-solving skills at the beginning of their studies.[5] Within the course, we deal with the basics of number theory, equations, inequalities, functions, and representations. Word problems are also included.

The students were informed and consented that they would be participants in the long-term research aimed at improving and streamlining the education of pre-service mathematics teachers.

3.2. **Methods of data collection**

At the beginning of the course, we solved several word problems belonging to school mathematics. The expected school strategy for solving them was equational (one linear equation with one unknown or a system of two linear equations with two unknowns). In addition to this strategy, other solving strategies were also included (trial and error, estimation–verification–correction, systematic experimentation, drawing a picture, and logical reasoning), till the given problem would be exhausted[6]. In our opinion, students had no or little experience with these strategies in problem-solving. But due to their simplicity and natural use, there was no difficulty for using them. Students found drawing a picture and estimation-verification-correction to be very surprising but suitable for the solution of word problems. Fig. 2 presents the solutions by drawing a picture for two selected and solved problems.

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[4] This case study has been planned as a part of longitudinal action research aimed at improving the problem-solving skills of pre-service mathematics teachers and their preparation.

[5] Students come to study from different types of upper secondary schools, with different levels of knowledge and skills in mathematics.

[6] As we mentioned above about the fourth phase of problem-solving.

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As we expected based on previous experience, drawing a picture evoked an “Aha!” effect for some students.[7] For this reason, we focus more on this strategy as well.

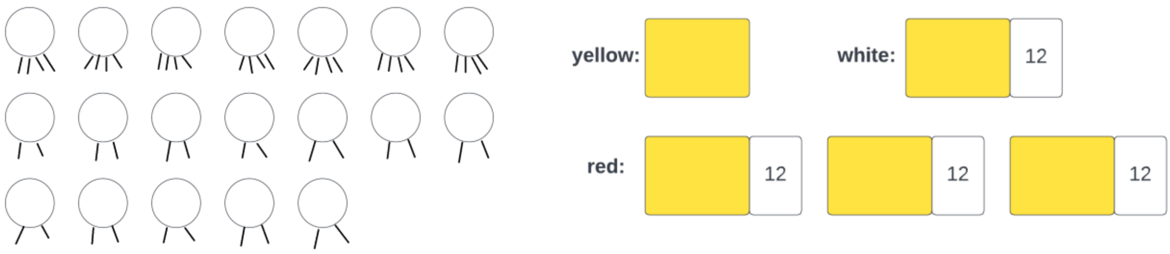

Figure 2. Drawing a picture for the ‘heads and legs problem’ (The children went for a walk with the dogs from the shelter. There were 19 heads and 52 legs. How many children and how many dogs were there?) and ‘tulips problem’ (Katka grows tulips in the garden. She planted overall 148 tulip bulbs, of which white ones were 12 more than yellow ones and red ones were three times more than whites. How many white tulip bulbs were there, how many yellow ones, and how many red ones?) (source: authors)

After calculating each word problem with several strategies, we posed together new problems, the solution of which would be based on the same (or similar) principle. In this fourth phase, we no longer focused on the generalization of the problem.[8]

After six lessons, students were asked to solve 3 word problems from school mathematics (Fig. 3, 4, 5), each of them with three different strategies. All the given problems can be solved in school algorithms. For all three problems, the students had to pose another word problem that would have a solution based on the same principle as were for the given word problems.

The first paragraph after a heading is not indented (Bodytext style).

After a few weeks of the semester, students were asked to fill out a questionnaire. Based on the answers, we wanted to find out how students relate to individual strategies and problem-posing.

The students’ solutions to the given word problems are qualitatively processed. The results of observation are also included in the paper. We are aware that one of the authors leads the given course and this may affect the subjective results.

[7] We really like this moment of enlightenment in education, and we think that students feel the challenge at that moment as a manifestation of internal motivation.

[8] We plan it later when students improve students’ solving skills.

3.3. Research tools

In this part, the given word problems and their characteristics are briefly described, and the questions from the questionnaire are written.

> Today is Jan and Alice's birthday. Jan is 5 years younger than twice Alice's age. Ten years ago, they were 65 years old together. How old is Jan today?

Figure 3. 'Age problem' (source: external exam, 2019[9])

The 'age problem' (Fig. 3) was aimed at testing the ability to construct and solve a linear equation with one unknown (or a system of two equations with two unknowns).[10]

> In the recipe for lečo, it is recommended to mix tomatoes, peppers, and onions in a ratio of 4:3:1. The cook had already prepared onions and peppers, and the onions were five kilograms less than the peppers. How many kilograms of tomatoes will be needed according to this recipe?

Figure 4. 'Lečo[11] problem' (source: testing T9, 2017[12])

The tested skill in the 'lečo problem' (Fig. 4) was the use of ratio in solving a word problem. The expected solution should have been arithmetic, because the difference between the weight of pepper and the weight of onion is two parts weighing 5 kg (so one part is weighing 2.5 kg).[13]

> 60 kg of oranges in four- and five-kilogram packages were brought to the store. How many four-kilogram and how many five-kilogram packages were there?

Figure 5. 'Orange problem' (source: authors)

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[9] The 'age problem' came from an external form of mathematics exam that students take at the end of upper secondary school. This problem was assigned to the tested thematic unit Basics of mathematics. The success rate was 77,3% so this problem was easy for students.

[10] This problem can be classified as a standard one from school mathematics, which strongly evokes the use of a variable (one or more) and an equational strategy. By the term standard problem, we call a problem from school mathematics for which well-known formulas or procedures (algorithms) are used to solve. A non-standard problem is then a problem for which the solver does not know formulas or procedures (algorithms).

[11] Lečo (in Hungarian lecsó) is a traditional Hungarian vegetable food that traditionally contains peppers, tomato, onion, salt, and ground sweet and/or hot pepper as a base recipe (it is often supplemented with eggs, bacon, or sausage). It is also considered to be traditional food in Slovak and Czech cuisine.

[12] The 'lečo' problem came from wide-nation Slovak testing T9 that students take at the end of lower secondary school. The success rate was 26,2%, The percentage of students who did not solve the task was up to 21%. The high failure rate of students was the reason for including this problem in the training of preservice teachers. (Csachová, Jurečková, Tkačik, 2021)

[13] Compared to other problems that are solved on the topic of ratio, we consider this problem to be non-standard. Most of the practiced problems are formulated in such a way as to divide the whole into parts in a certain ratio.

The 'orange problem' (Fig. 5) can be solved using the least common multiple.[14] This problem is also interesting because it has not only one correct result but four. (However, this fact does not follow from the assignment, there is no instruction that would guide the solver to find all the solutions.) In addition to the results of 10 four-kilogram and 4 five-kilogram packages, and 5 four-kilogram and 8 five-kilogram packages, there are other two "extreme" results. The first result is 15 four-kilogram packages (if they did not bring any five-kilogram packages) and the second one is 12 five-kilogram packages (if they did not bring any four-kilogram packages).

Any of these three problems are usually solved in the arithmetic or algebraic way, not in a graphic way. Their possible solutions as drawing a picture are in Fig. 4.

In the questionnaire, there were 6 items. In five of them, students had to write: 1. strategies they knew from mathematics lessons from primary and secondary school, 2. strategies they did not know at all, 3. strategies they consider useful and suitable for solving word problems, 4. strategies they will/want/plan or 5. they will not/do not want/do not plan to use for solving word problems. If they wanted, they could also state a reason for each item. In item number 6, they had to say whether they consider problem-posing to be difficult or easy and useful or useless.

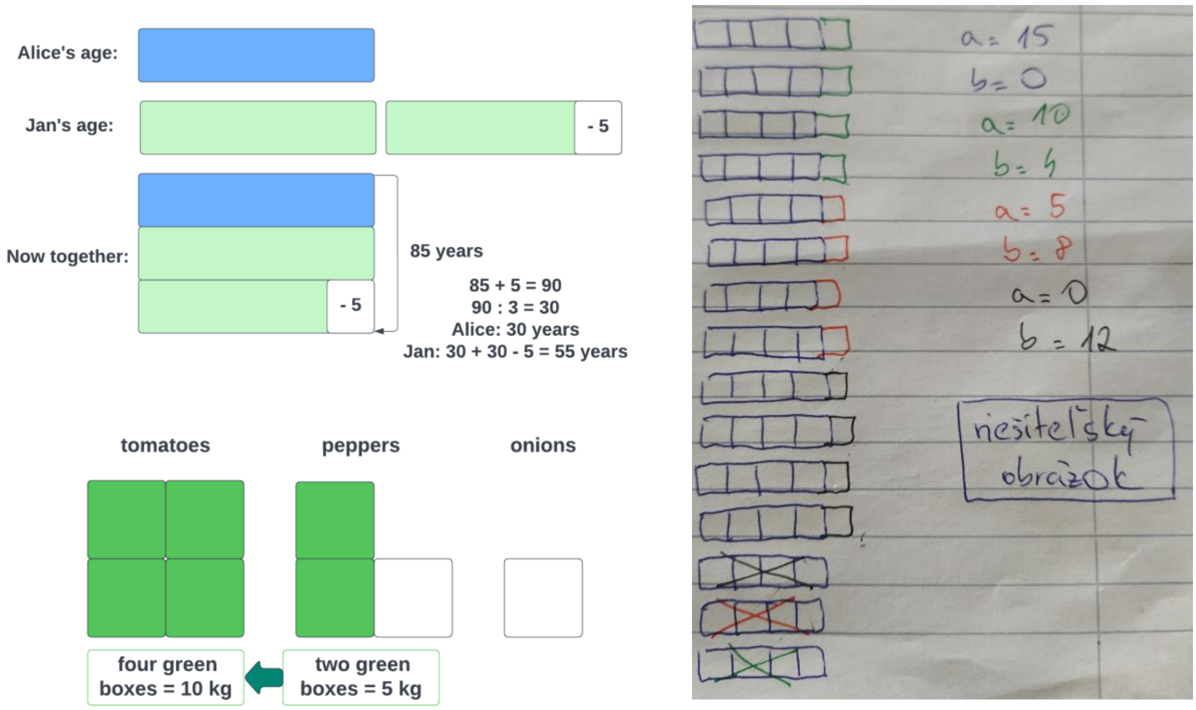

Figure 3. Solutions of the 'age problem', 'lečo problem', and 'orange problem' with drawing a picture strategy (source: authors, original pictures came from solutions of magister-level students)

[14] The mathematics teacher told us that such problems are solved in mathematics in lower secondary school. But the solving strategy depends on the pupils themselves, there are not presented with the specific strategy. Therefore, we can consider this problem as non-standard.

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4. Results paper

At the beginning of this chapter, we present solutions from our students, as well as new problems posed by them. In the end, there is an overview of their opinions about learned and used solving strategies. We briefly compare them with our observations during our lessons.

4.1. ‘Age problem’

Nine students solved the ‘age problem’ with three strategies, and one student used only two strategies. The assumption of choosing the equational strategy as the first was met, as many as nine students chose this expected school strategy as the first, and for one student it was the second choice (logical reasoning was the first). Other used strategies were “newly learned”: estimation-verification-correction, systematic experimentation, and logical reasoning. The students solved the problem partially or completely with pictures only in three cases, e. g. Fig. 4.

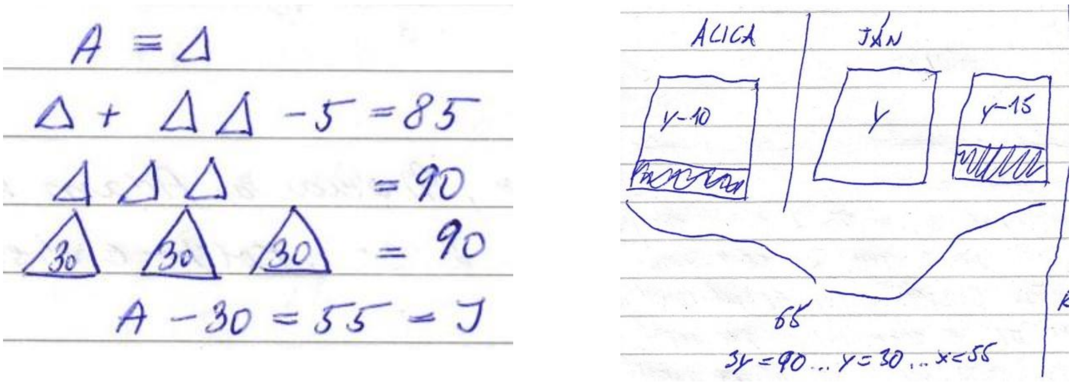

Figure 4. Drawing a picture (source: authors, students’ solutions)

After the solution of the problem with different strategies, the students were asked to pose a new problem, which solution would be on the same principle as the original one. Five newly posed word problems were devoted to the calculation of years. Some of them copied the assignment of the given problem, such as: *The tortoise Dorota is 158 years old, and her daughter Little Dorota is 71 years old. How many years ago was Dorota 30 times older than Little Dorota?* Three newly formulated word problems were interesting, and in each of them, there is some information connecting with ‘time’ (years). But this time information does not affect the result in any of them. The first of these three newly posed problems was the following one: *Modern families have two cars. Today, the man has 50,000 km less on his car than three times the mileage on his wife’s car. Ten years ago, they had driven a total of 65,000 km. How many kilometres has the man driven today?* The second one was: *Jan and Alica are children of rich parents. Jan has 5,000 euros less than three times Alice’s money. Five years ago, they had got 165,000 euros together. How many euros has Jan got today?*

The last one was very similar to the previous one. The problem was about rich parents paying for the birthday party of their children. There was information including the time leap, but this information was not important for solving the problem.

4.2. ‘Lečo problem’

Eight students solved this problem with three strategies, one student with two, and one student with only one strategy. For this problem, we assumed logical reasoning with the ratio as the first solving strategy (on the arithmetic base). However, the equational strategy was the first option for five students, for three students it was a drawing a picture strategy, and only two students solved this problem using a ratio.

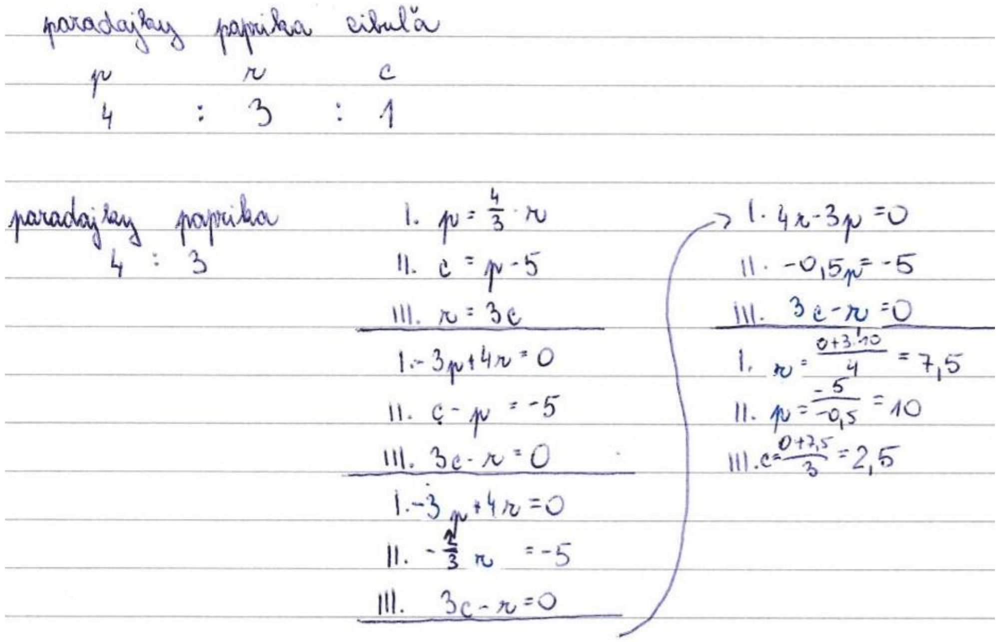

Figure 5. Equational strategy (3 unknowns: *p* – the weight of tomatoes, *r* – the weight of peppers, *c* – the weight of onions, the relationships between them are expressed by a ratio) (source: authors, student's solution)

When we look at all mentioned strategies in this problem's solution, all ten students used an equation strategy. In two cases it was even a set of three equations with three unknowns and in one case a set of two equations with two unknowns. However, the principles of the compilation of equations were different. While each of the three equations was constructed on the weight proportions of individual types of vegetables, where unknowns were the weights of vegetables – Fig. 5, in the solution in Fig. 6 the first unknown *x* represented the weight of onions, and the second unknown was the total weight of all ingredients.

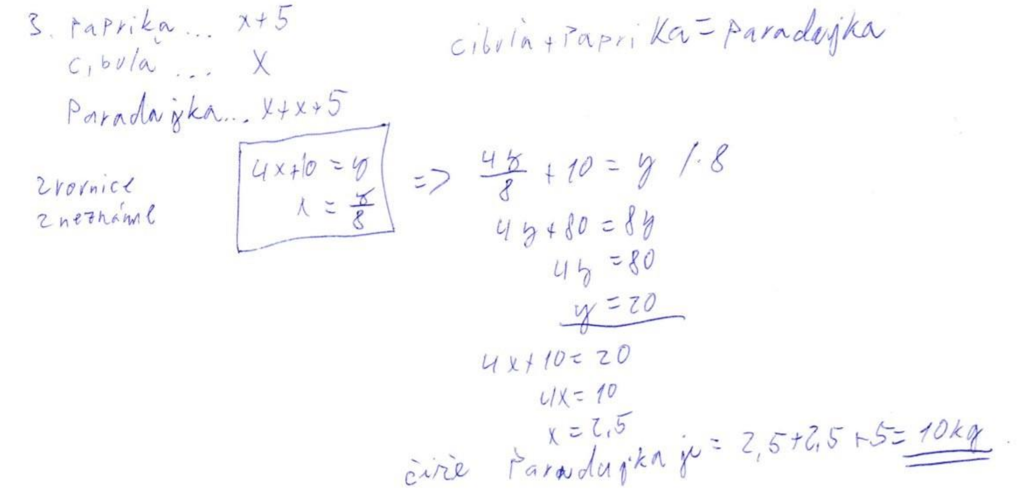

Figure 6. Equational strategy (2 unknowns: *x* – the weight of onions, *y* – the total weight of ingredients) (source: authors, student's solution)

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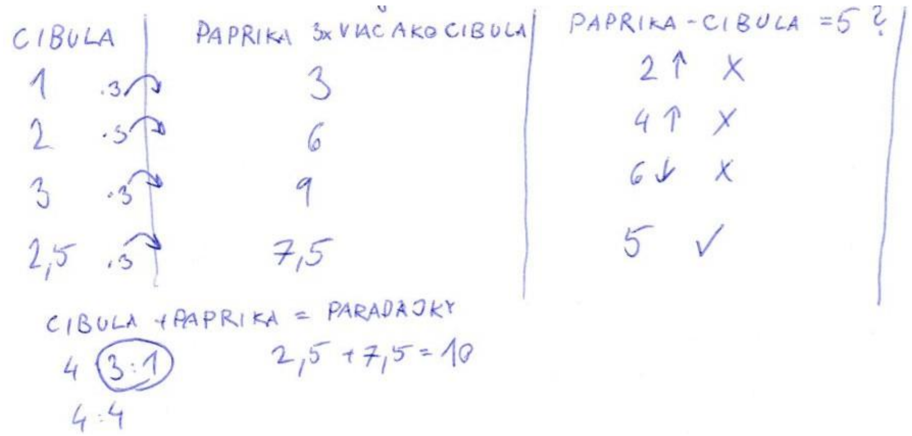

Figure 7. Systematic experimentation: in the first column, there is the weight of onions, in the second one there is the weight of peppers (it has to be three times more than the weight of onions), and numbers in the third column represent the difference between the weight of peppers and the weight of onions (it should be 5 kilograms, darts by numbers marked that the difference should be more or less the result) (source: authors, student's solution)

Other used strategies were drawing a picture (seven students), logical reasoning (six students), trial and error (two students), and estimation-verification-correction (two students). In Fig. 7 there is a solution based on the strategy of systematic experimentation.

Among the newly posed word problems, there are 7 word problems whose structure copies the assignment of the given problem. As an example, we present: *The ratios of the lengths of the sides of the triangle are 3:4:5 (a:b:c), with side b being 5 cm shorter than side a. What is the length of side c?*[15] Another 2 word problems have the same structure as standard ratio problems from school mathematics, e. g. *Three friends Peter, Mária, and Adam divide 180 euros in the ratio 2:3:1. How many euros will Maria receive?* One student posed a word problem, whose context was cooking, but the essence was direct proportionality. (However, the student posed assignments of the other 2 word problems in accordance with the requirements.)

4.3. 'Orange problem'

As was mentioned above, this word problem can be considered non-standard. We assumed that the first strategy used would be trial and error or estimation-verification-correction.

In this problem, seven students solved the problem with three strategies, two students each used two strategies, and one student only one strategy. Drawing a picture was the most frequently used of all strategies (seven students) – the examples are shown in Fig. 9, 10. For the solution, six students used logical reasoning, and five students used estimation-verification-correction. Systematic experimentation occurred twice and trial and error once. But only three students wrote all four possible results.

[15] The problem is also listed with a mistake in the assignment because the ratio a:b:c = 3:4:5 implies that side b cannot be shorter than side a. (Later, this mistake was corrected by the student.)

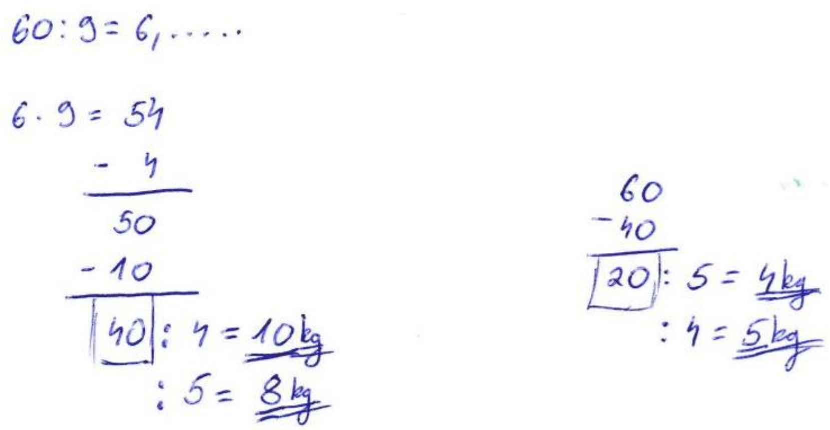

Figure 8. Logical reasoning (in written arithmetical form): the student first found out how many 9-kilogram packages it would be possible to bring (if the numbers of both types of packages were the same). But the solution was based on finding numbers that are divisible by 4 and 5 and their sum is 60. (source: authors, student's solution)

As the first strategy, students chose estimation-verification-correction in four cases, and the equational strategy in two cases. Other strategies (drawing a picture, trial and error, systematic experiments, and logical reasoning) occurred in one case each.

When creating a word problem with the same solution principle as the solution to the 'orange problem', the students showed the most creativity. The original problem was formulated clearly and distinctly, such as many situations from real life can be described similarly. Thanks to this, eight students posed new word problems, which were correctly formulated and the "numbers" appearing in their assignment were correct (i. e. the problem had a solution). We present two of them all. First problem: *There were a total of 72 students on the trip. They were divided into groups of 8 or 9 members. How many groups of 8 and 9 were there?* As the second one, we present a word problem that is suitable for developing financial literacy*: I told my daughter that I paid 60 euros in the store today. However, I only had 2-euro coins and 5-euro notes. How many 2-euro coins and how many 5-euro notes could I pay?*

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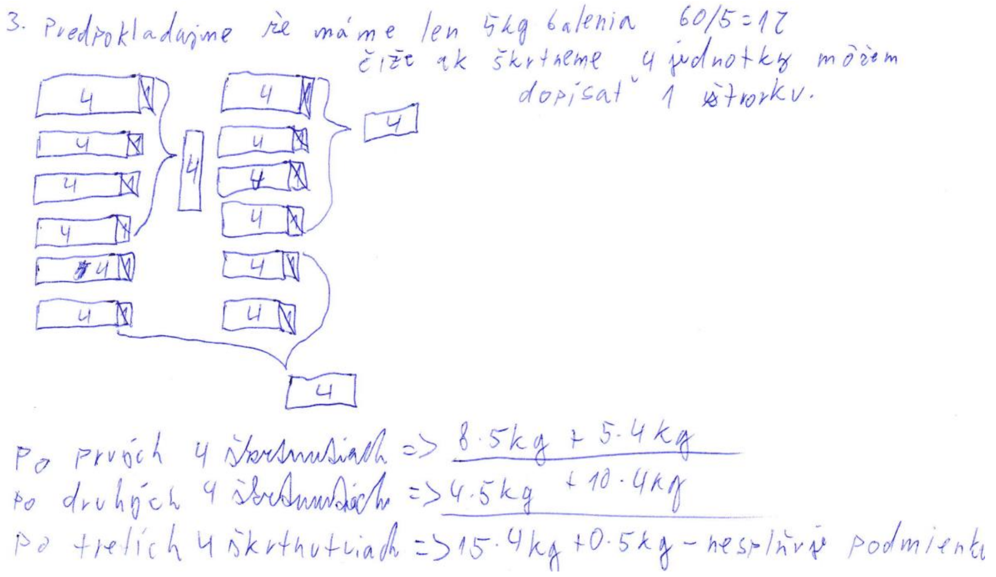

Figure 9. Drawing a picture I: the solution is based on the assumption that we only have 12 five-kilogram packages, from which we gradually take 1 kilogram of oranges and gradually assemble four-kilogram packages (the same principle as in Fig. 3) (source: authors, student's solution)

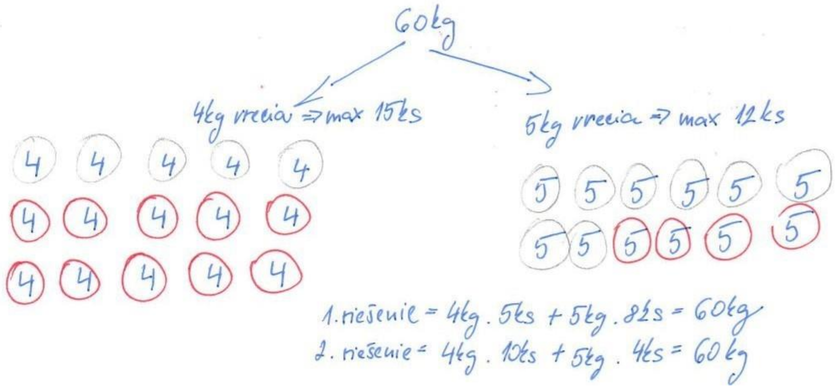

Figure 10. Drawing a picture II: the number of fours on one side and the number of fives on the other side correspond to the "extreme" results of the task; the student then tried to combine packages from one side and the other so that the sum of the weights was 60 kg (source: authors, student's solution)

4.4. Students' opinions on strategies

After the students' solutions and posing new problems, we tried to discover students' opinions on the strategies they used and might use.

Table 1 summarizes the numbers of students' answers to the questions from the questionnaire: known/unknown (which of the solving strategies they knew/did not know from teaching mathematics in primary and secondary school), useful and suitable (for solving word problems), and to be used /not to be used (which strategies they will / will not use for solving word problems).

strategy for word problems solution	known from math. lessons	unknown from math. lessons	useful and suitable	plan to be used	plan not to be used
trial and error	5	1	2	4	2
est.-verif.-correct.	4	2	2	-	2
systematic exper.	2	4	4	1	2
logical reasoning	5	1	5	6	1
drawing a picture	5	2	8	5	2
equational str.	8	-	8	7	1

Tab. 1 The number of students' answers to the questionnaire (source: authors)

Unsurprisingly, the most well-known solving strategy from teaching mathematics at primary and secondary school was the equation (eight students), trial and error, logical reasoning, and drawing a picture strategy were mentioned in five cases. The least known strategy was systematic experimentation (four cases). When solving word problems, students plan to mainly use the strategies equation (seven cases), logical reasoning (six cases) and drawing a picture (five cases). They consider them "irrefutable and more mathematical than the first three". They think that trial and error, estimation-verification-correction, and systematic experimentation are applicable "perhaps in extreme cases" because "they didn't suit me personally" or, for example, systematic experimentation "seems very time-consuming... it takes up time that I can use for another problem".

Our students do not yet fully understand the picture in the solution as a strategy, but as something that "helps us imagine and better understand the given problem" or "sometimes I can't move without a picture, a better understanding of the task when I have drawn what we know and what we do not".

Two students stated a positive approach to all mentioned solving strategies: "I will probably use all (strategies)... we can use different strategies for different problems." and "(I would like) students to know what all the methods (solutions) are... so that they don't give up and try other methods if something doesn't work out for them".

Compared to the very positive reactions to the various problem-solving strategies from the teachers in practice, our students have so far insisted on the equation strategy as a certainty. The students said that "it is the fastest way to get to the result" or "when I read a word problem, I automatically think about an equation", "equations are a great tool"[16], or "(equations) a simple and proven type of solution that is sufficient for every teacher".

This opinion was also manifested, for example, when solving problems from the mathematic olympiad (for age group: 11-13-year-old pupils). The students reached for equations as a certain strategy.[17]

However, in the solutions to other problems in our next lessons, drawing a picture strategy began to appear more often, as well as a bolder approach to the use of trial and error or estimation-verification-correction.

[16] Even though this student stated: "(trial and error) this is the first thing that goes through my head".
[17] In Slovakia, equations are included only at the end of lower secondary school (14-15-year-old pupils).

5. Conclusion

The paper presents the results of a case study with aim to improve students' problem-solving competencies especially thanks to the fourth phase 'looking back' in the word problem-solving process. This, among other things, consists in finding other ways = solving strategies and posing new problems, which solution is based on the same principle as the solution of the original problem.

In addition to the strategies students knew (e. g. equational strategy or logical reasoning), they also used strategies less familiar to them (which they learnt) to solve problems. But we were very surprised (when compared with previous pre-service teachers and experienced mathematics teachers) that the students recognized the importance of, for example, drawing a picture, trial and error, or estimation-verification-correction, but still preferred the equational strategy. However, we will continue with this approach to solving problems.

When searching for different strategies, students are highly influenced by the result from the previous strategy. For example, if the third strategy was trial and error or estimation-verification-correction, they had trouble "forgetting" the result and trying to "fit" those values so that it works out for them.

Problem-posing can reflect the understanding of solutions to the given problems. It seems that those students who know how to solve problems also know how to make a word problem with a similar solution. However, we realize that the answer to research question 3 is not as simple as we assumed. New posed problems require a more precise analysis based also on the atomic analysis of individual assignments and comparison with the assignments of the original word problems. The first 'age problem' was easily solved with the equational strategy, but problem-posing was more difficult for students. While for the 'orange problem', the students did not have a clear school strategy, and yet they posed adequate new problems. Nevertheless, we realize that the level of problem-solving can be correlated with the level of posing new problems. From this point of view, this contribution can serve as preliminary research for larger research that will investigate the phenomena just mentioned.

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